AIAA 80-1708R

Control Logic for Parameter Insensitivity and Disturbance Attenuation

Arie Ashkenazi*
Ministry of Defense, Haifa, Israel
and
Arthur E. Bryson Jr.†
Stanford University, Stanford, Calif.

An approach to the synthesis of control logic that is both insensitive to system parameters and attenuates response to input disturbances (sometimes called robust control logic) is presented and is applied to a simple system with an uncertain vibration frequency. A conclusion drawn from this study is that the lower the order of the compensator dynamics, the lower the sensitivity of the closed-loop system performance to parameter variations. It follows that estimated-state feedback is undesirable when there are large uncertainties in the system parameters. However, quadratic performance indices are minimized with estimated-state feedback, so the designer must trade off parameter insensitivity against disturbance attenuation. This is done here by minimizing the expected value of a sum of quadratic performance indices, each one of which is evaluated with different values of the system parameters; the decision variables in the minimization are parameters in a compensator of specified order that is less than or equal to the order of the system model.

Nomenclature

A	= compensator dynamic matrix
a	= vector of system parameters
B	= compensator input matrix
C	= compensator output matrix
D	= gain matrix from measurements to controls
E( )	= expectation operator
F	= system dynamic matrix
G	= control input matrix
H	= system measurement matrix
J	= performance index
k	= spring constant
N	= noise input matrix of the augmented system
P( )	= probability of ( )
p	= vector of compensator parameters
Q	= disturbance spectral density
R	= measurement noise spectral density
S	= augmented system dynamic matrix
tr( )	= trace of ( )
u	= system control vector
v	= measurement white noise with zero mean
w	= white noise disturbance vector with zero mean
X	= state covariance matrix of augmented system
x	= system state vector
y	= measurement vector
z	= compensator state vector
Γ	= system disturbance input matrix
$\Delta(\ )$	= deviation of ( ) from its nominal value
Λ	= measurement noise input matrix
$(\ )_i$	= value of ( ) at point i in the system parameter space
$(\ )^T$	= transpose of ( )

Presented as Paper 80-1708 at the AIAA Guidance and Control Conference, Danvers, Mass., Aug. 11-13, 1980; submitted Oct. 23, 1980; revision received Sept. 24, 1981.

*Research Engineer.
†Fellow AIAA.

Introduction

THERE is a wide range of performance objectives for feedback control systems. The minimum requirement is to establish stability of the nominal system. Beyond that, some of the common objectives are listed below:

1) parameter insensitivity—to establish stability of the system for a range of off-nominal values of the system parameters;

2) disturbance attenuation—to reduce the system response to disturbances to an acceptable level;

3) command response—to make the system respond to commands with satisfactory speed and accuracy; and

4) tracking response—to make the system track the outputs of another system with satisfactory speed and accuracy.

These objectives may conflict with each other. In particular, parameter insensitivity may conflict with the other three objectives. This has become increasingly apparent in the last two decades as optimization techniques have been developed. The performance criteria used in linear quadratic Gaussian synthesis have focused on objectives 2, 3, and 4 (Ref. 1) and have assumed the system parameters to be known exactly. For lightly damped oscillatory systems with only one measurement, the resulting estimated-state feedback controller often turns out to be so sensitive to changes in system parameters that it is not useful[2]; some performance has to be sacrificed to desensitize the controller.

Uncertainty in the knowledge of system parameters is one source of parameter variations. Another source is predictable variations such as those that occur in the stability derivatives of an airplane as it changes speed, altitude, or angle of attack. The subject of controller design for disturbance attenuation has been treated many times before but in most cases full state feedback was assumed.[3-8] In most practical problems the full state is not known, and the use of state estimates based on inaccurate models is the source of increased parameter sensitivity; in some cases this sensitivity results in failure of the linear-quadratic-regulator method. Levine and others[9,10] have treated the problem of output feedback for deterministic systems; Hadass[11] has worked out the problem of output feedback in systems with uncertain parameters. However, his solution is confined to small parameter variations using explicit state estimations.

In this paper, an algorithm is presented for designing dynamic compensators that are guaranteed to give stable closed-loop performance for a specified range of certain plant parameters. For more details on the algorithm and several more realistic examples, the reader is referred to Ref. 12.

The Parameter-Insensitive Disturbance-Attenuating Controller

We shall represent the plant (system to be controlled) by

$$\dot{x}=F(a)x+G(a)u+\Gamma(a)w \tag{1}$$

where a is a vector of system parameters.

The measured outputs form a vector y, where

$$y=H(a)x+\Lambda v \tag{2}$$

and v is the white noise disturbance vector with spectral density R.

The controller will be of the form

$$u=C(p)z+D(p)y \tag{3}$$

$$\dot{z}=A(p)z+B(p)y \tag{4}$$

where the dim (z) is specified.

The controller parameters, p, are to be chosen to minimize the steady-state value of the performance index

$$J=\sum_{i=1}^{k}P_iJ_i(a_i,p) \tag{5}$$

where

$$J_i(a_i,p)=\mathrm{E}[x^TW_xx+u^TW_cu]_{a=a_i} \tag{6}$$

and the expectation operator E is applied over the distributions of w and v.

A Block-Diagonal Form for the Controller

For the minimization of Eq. (5) to be unique, the structure of the controller matrices A, B, and C must be such that the dimension of p is the smallest possible integer for the specified controller order[13] and the specified dimensions of u and y; namely,

$$\dim(p)=[\dim(u)+\dim(y)]\dim(z)+[\dim(u)\cdot\dim(y)] \tag{7}$$

We have used the block diagonal form introduced by Martin,[13] since it provides minimal sensitivity of the compensator eigenvalues to inaccuracies in its parameters, and handles complex as well as real eigenvalues using only real arithmetic.

$$A=\left[\begin{array}{c|c|c} A_1 & 0 & \\ \hline 0 & A_2 & \\ \hline & & \ddots \end{array}\right]\quad B=\begin{bmatrix} B_1\\ B_2\\ \vdots\end{bmatrix}\quad C=[C_1,C_2,\ldots] \tag{8}$$

where the A_i are 2×2 blocks of the form

$$A_i=\begin{bmatrix}0 & 1\\ a_{1_i} & a_{2_i}\end{bmatrix} \tag{9}$$

except when dim (z) is odd, in which case the last A_i is simply a real number. Also, the C_i has two columns

$$C_i=\begin{bmatrix}0 & 1\\ C_{21}^{(i)} & C_{22}^{(i)}\\ \vdots & \vdots\end{bmatrix} \tag{10}$$

except when dim (z) is odd, in which case the last C_i has only one column of the form

$$C_i=\begin{bmatrix}1\\ C_2^{(i)}\\ C_3^{(i)}\\ \vdots\end{bmatrix} \tag{11}$$

The form, Eqs. (8-11), cannot represent controllers with exactly repeated eigenvalues, but this does not appear to be a serious limitation here.

Determination of J_i

From here on, we treat systems with measurements contaminated by noise. To avoid control commands generated by noise we place $D=0$. In cases where actuator dynamics are significant, the decision on D is based on noise bandwidth relative to actuator bandwidth.

If we define

$$X\triangleq\mathrm{E}\left[\begin{array}{c|c} xx^T & xz^T\\ \hline zx^T & zz^T\end{array}\right] \tag{12}$$

then the steady-state value of X are given by the solution of the Lyapunov equation (see, for example, Ref. 1):

$$0\triangleq SX+XS^T+NVN^T \tag{13}$$

where

$$S\triangleq\left[\begin{array}{c|c} F & GC\\ \hline BH & A\end{array}\right]\quad N\triangleq\left[\begin{array}{c|c} \Gamma & 0\\ \hline 0 & B\Lambda\end{array}\right]\quad V\triangleq\left[\begin{array}{c|c} Q & 0\\ \hline 0 & R\end{array}\right]$$

and stable closed-loop behavior has been assumed. The value of J_i is then given by

$$J_i=\mathrm{tr}\left\{\left[\begin{array}{c|c} W_x & 0\\ \hline 0 & C^TW_cC\end{array}\right]\cdot X\right\} \tag{14}$$

Algorithm for Determining the Parameter-Insensitive Disturbance-Attenuating (PIDA) Controller

Minimization of Eq. (5) by choice of p is a constrained optimization problem (or nonlinear programming problem; see, for example, Ref. 1). A Hamiltonian function may be defined as

$$H=\sum_{i=1}^{k}\left\{P_iJ_i(a_i,p)+\mathrm{tr}(\Lambda_i[S_iX_i+X_iS_i^T+N_iVN_i^T])\right\} \tag{15}$$

where the subscript i denotes evaluation with parameters a_i, and Λ_i is a Lagrange multiplier matrix. Necessary conditions for minimization of Eq. (5) by choice of p are then Eq. (13) with k different values of a_i, plus

$$0=\frac{\partial H}{\partial X_i}=P_iW+\Lambda_iS_i+S_i^T\lambda_i \tag{16}$$

$$0=\frac{\partial H}{\partial p}=\sum_{i=1}^{k}\left\{\left[P_i\frac{\partial w}{\partial p}+\Lambda_i\frac{\partial S_i}{\partial p}+\frac{\partial S_i^T}{\partial p}\Lambda_i\right]X_i+\frac{\partial N_i}{\partial p}VN_i^T+N_iV\frac{\partial N_i^T}{\partial p}\right\} \tag{17}$$

where

$$W=\left[\begin{array}{c|c} W_x & 0\\ \hline 0 & CW_cC^T\end{array}\right]$$

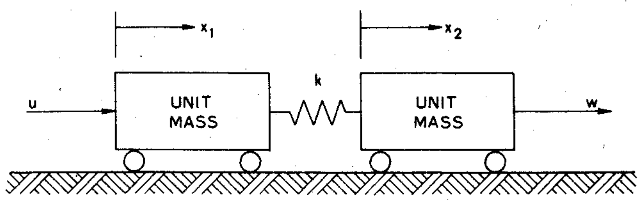

Fig. 1 Physical model of example.

These are coupled equations. However, assuming values of p, Eq. (13) determines X_i; Eq. (16) determines Λ_i; and Eq. (17) determines values of p that give a stationary value of J in Eq. (5).

A quasi-Newton method[13] was used to solve the nonlinear programming problem described above.

The new contribution here is to include a compensator of lower order than the system, along with random disturbances. The idea of using a sum of quadratic performance indices, each one evaluated at different values of the system parameters is due to Vinkler.[5]

Example 1: An Elastic System of Fourth Order with One Uncertain Parameter

We consider the following fourth-order system with one uncertain parameter, first with a second-order controller, then with a fourth-order controller:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 1 \\ k & 0 & -k & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w \tag{18}$$

$$y = x_2 + v$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} y$$

$$u = [0,\ 1]\,[z_1, z_2]^T \tag{19a}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & \\ a_1 & a_2 & & \\ \hline 0 & & 0 & 1 \\ & & a_3 & a_4 \end{array}\right] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} y \tag{19b}$$

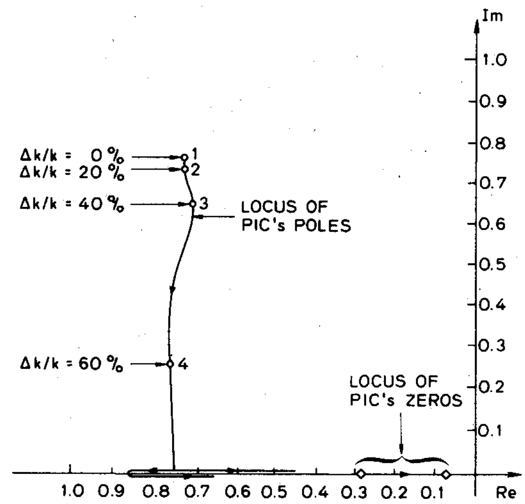

Fig. 2 Locus of poles and zeros of compensator transfer function $u(s)/y(s)$ vs anticipated parameter variation Δk for the second-order parameter-insensitive controller.

$$u = [0,1,0,1] \cdot [z_1, z_2, z_3, z_4]^T$$

Equations (18) arise from the physical system shown in Fig. 1, namely, two unit masses connected by a spring with an uncertain spring constant k acted on by a control force u and a random disturbance force w; y is a measurement of the position of the right mass. Equations (19a) and (19b) are the state space representation of a second- and fourth-order controller, respectively.

As performance index we shall use

$$J = P_1 J_1 + P_2 J_2 + P_3 J_3 \tag{20}$$

where k_1 is the nominal k; $k_2 = k_1 + \Delta k$; $k_3 = k_1 - \Delta k$; P_i equals the probability of $k = k_i$; and

$$J = \mathrm{E}[y^2 W_y + u^2 W_c] \tag{21}$$

Here we have taken

$$k_1 = 1 \qquad P_1 = 10/11 \qquad P_2 = 1/22 \qquad P_3 = 1/22$$

$$W_y = w_c = 1 \qquad q = r = 0.01 \tag{22}$$

Figures 2 and 3 show the loci of zeros and poles for the optimal second- and fourth-order parameter insensitive controllers (PIC's) vs specified Δk.

The optimal fourth-order controllers for $\Delta k = 0$ and 0.2 have pole-zero pairs quite close to the oscillatory mode of the plant; this is called "notch compensation" since the Bode magnitude plot has a notch in it near the resonant frequency. However, for $\Delta k = 0.4$, these pole-zero pairs have moved quite far away from the nominal plant oscillatory poles, and for $\Delta k = 0.8$, these compensator poles have moved very far out on the negative real axis, while the zeros have started to come back toward the $j\omega$ axis. The other two poles and the real zero are quite close to the positions of the two poles and the real zero of the optimal second-order compensator, and the steady-state compensator gains,

$$k_{ss} = \lim_{s \to 0} \, [u(s)/y(s)],$$

are nearly the same for $\Delta k \geq 0.4$ (see Fig. 4).

Figure 5 shows loci of the closed-loop poles of the slowest mode vs the actual k, using the optimal second-order compensators designed for $\Delta k = 0$, 0.2, 0.4, 0.6, and 0.8. Figure 6 shows the same loci using the optimal fourth-order com-

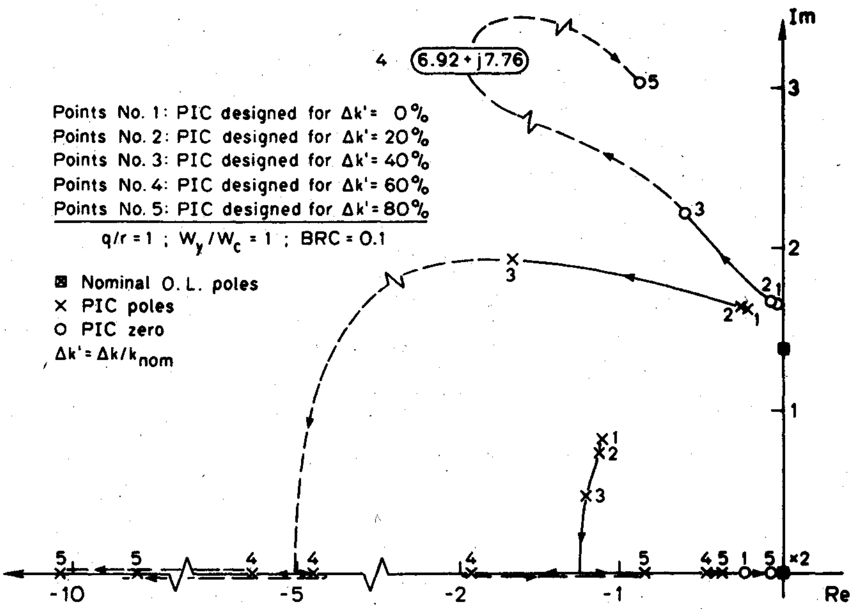

Fig. 3 Locus of poles and zeroes of compensator transfer function $u(s)/(y(s)$ vs anticipated parameter variation Δk for the fourth-order parameter-insensitive controller.

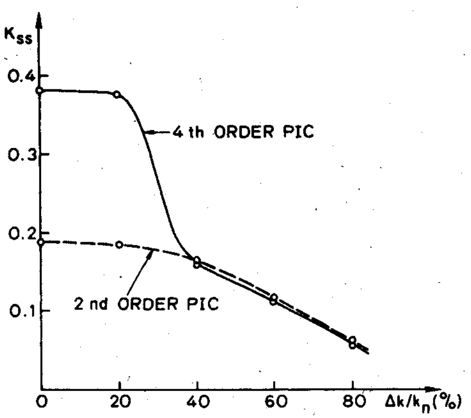

Fig. 4 Gain of compensator transfer function $u(s)/y(s)$ at zero frequency vs anticipated parameter variation Δk for second- and fourth-order parameter-insensitive controllers.

pensators. Note that the negative real part of the poles decreases (in the vicinity of the nominal open-loop pole) as we design for greater parameter insensitivity; i.e., we lose performance as we require more parameter insensitivity. Furthermore, the negative real parts of these closed-loop poles are larger with the fourth-order compensators than with the second-order compensators (in the specified uncertainty region).

Figure 7 shows the local performance index $J_l(k,p)$ as a function of actual k, using the optimal second-order compensators designed for $\Delta k = 0$, 0.2, 0.4, and 0.6. Figure 8 shows the same thing for the optimal fourth-order compensators. In the design regions, the fourth-order compensators always give better performance (lower performance index) than the second-order compensators. However, outside the design regions, the fourth-order compensators produce instability at smaller values of Δk than the second-order compensators (hence poorer performance outside the design regions).

Figure 9 compares the performance indices as a function of the parameter deviation $\Delta k/k$ using second- and fourth-order compensators. (ΔPI is the difference between the two performance indices.) From this figure it is apparent that the full-order compensator always results in a lower performance index; however, the differences are very small. This example exhibits the main properties of the PIC design algorithm.

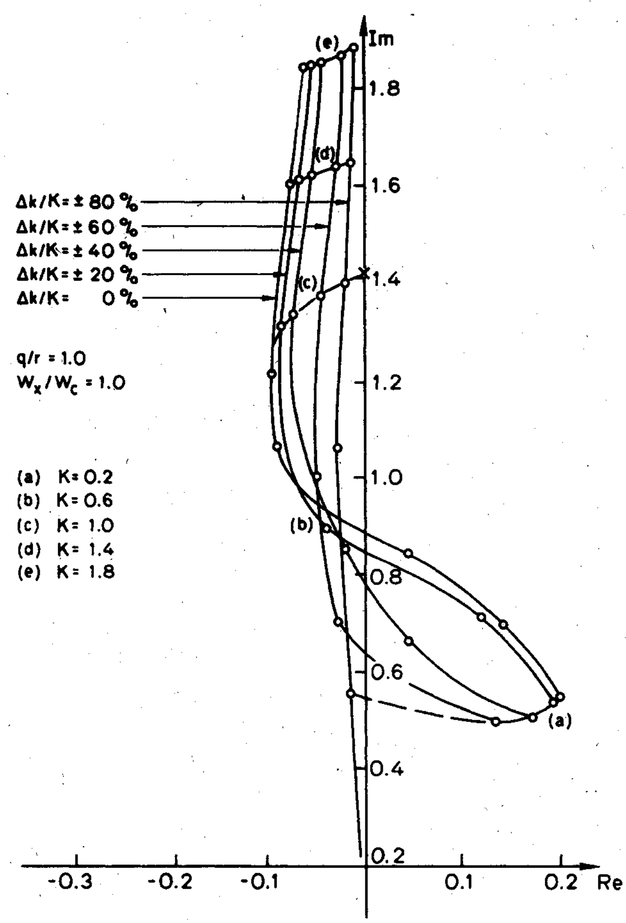

Fig. 5 Locus of the slowest closed-loop pole vs actual parameter k for several second-order controllers, each designed for a different anticipated Δk.

The lower-order compensator provides stable closed-loop operation over wider regions of parameter variations. This is a direct outcome of less tuning of the compensator to the system model. The higher the order of the compensator, the more tuning there is, which results in better performance indices but over a narrower region of parameter variations. This is symbolically shown in Fig. 10.

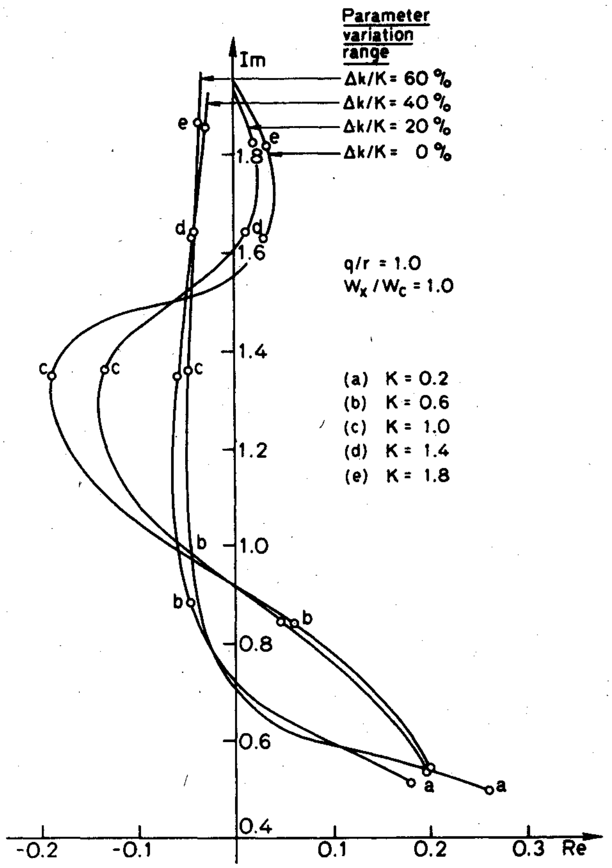

Fig. 6 **Locus of the slowest closed-loop pole vs actual parameter for several fourth-order controllers, each designed for a different anticipated Δk.**

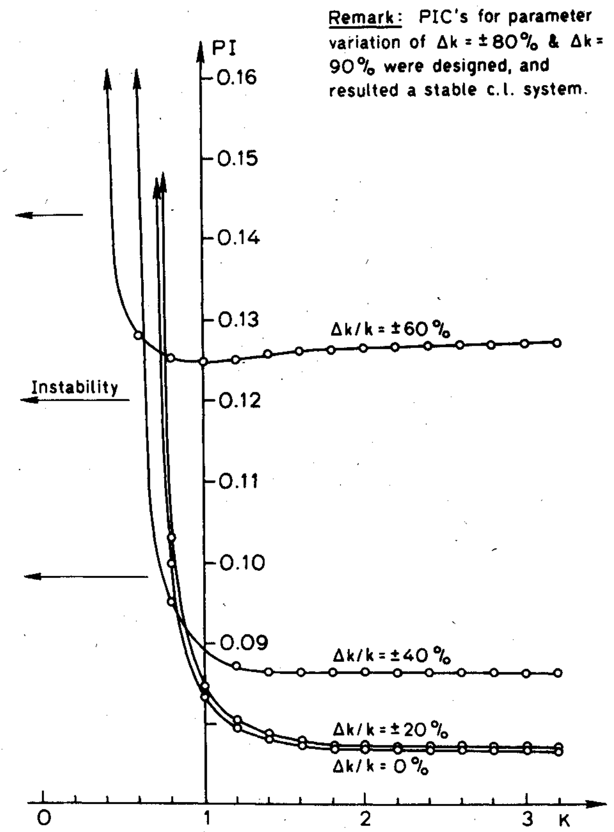

Fig. 7 **Local performance index vs actual parameter k for several second-order controllers, each designed for a different anticipated Δk.**

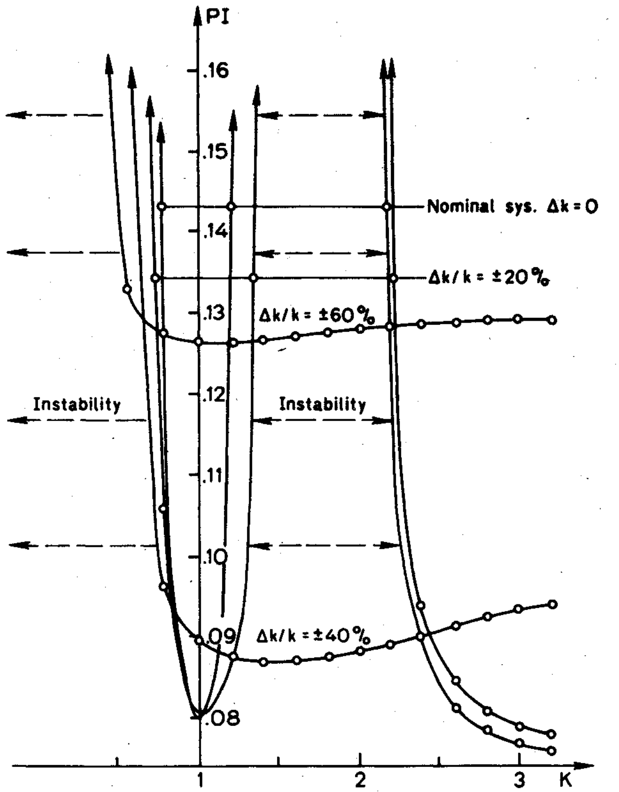

Fig. 8 **Local performance index vs parameter k for several fourth-order controllers, each designed for a different anticipated Δk.**

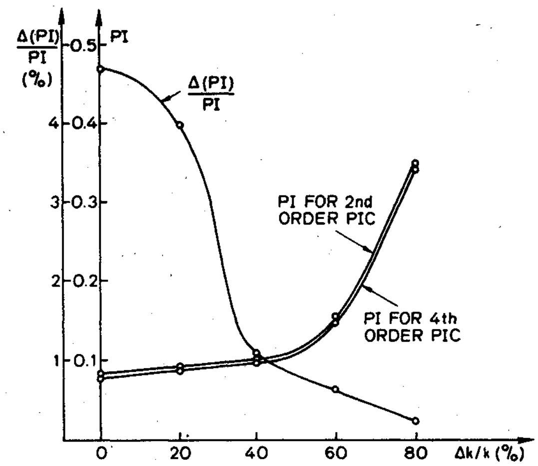

Fig. 9 **Total performance index vs anticipated parameter variation Δk for second- and fourth-order controllers.**

Conclusions

An algorithm has been presented for designing controllers of specified order to minimize response to zero-mean random disturbances with variations in some plant parameters specified. As expected, performance must be sacrificed to ensure stable operation with variations in the plant parameters. Furthermore, it appears from the examples we have done that the higher the order of the controller, the better the performance in the design range of the plant parameters, but the narrower the range of stable operation.

References

[1]Bryson, A.E. and Ho, Y.C., *Applied Optimal Control,* 2nd Ed., Halsted Press, New York, 1975.

[2]Doyle, J.C., "Robustness of Multiloop Linear Feedback Systems," *Proceedings of the 17th Conference on Decision and Control,* San Diego, Calif., Jan. 1979, pp. 12-18.

[3]Salmon, D.M., "Minimax Controller Design," *IEEE Transactions on Automatic Control,* Vol. AC-13, Aug. 1968, pp. 369-373.

[4]Vinkler, A. and Wood, L.J., "Comparison of Several Techniques for Designing Controllers of Uncertain Dynamic Systems," *Proceedings of the 17th IEEE Conference on Decision and Control,* San Diego, Calif., Jan. 1979, pp. 31-38.

[5]Vinkler, A., "Optimal Controller Design Methods for Linear Systems with Uncertain Parameters-Develop, Evaluation, and Comparison," Ph.D. Thesis, California Institute of Technology, Pasadena, Calif., April 1979.

[6]Harvey, C.A. and Pope, R.E., "Study of Synthesis Techniques for Insensitive Aircraft Control Systems," NASA CR 2803, April 1977.

[7]Harvey, C.A. and Pope, R.E., "Insensitive Control Technology Development," NASA CR 2947, Feb. 1978.

[8]Ly, U. and Cannon, R.H., "A Direct Method for Designing Robust Optimal Control Systems," *Proceedings of the AIAA Guidance and Control Conference,* Palo Alto, Calif., Aug. 1978, pp. 440-448.

[9]Levine, W.S. and Athans, M., "On Determination of Optimal Constant Output Feedback Gains for Multivariable Systems," *IEEE Transactions on Automatic Control,* Vol. AC-15, Feb. 1970, pp. 42-48.

[10]Levine, W.S., Johnson, T.L., and Athans, M., "Optimal Limited State Variable Feedback Controllers for Linear Systems," *IEEE Transactions on Automatic Control,* Vol. AC-16, Dec. 1971, pp. 785-792.

[11]Hadass, Z., "Design of State Feedback Controllers Including Sensitivity Reduction with Applications to Precision Pointing," Department of Aeronautics and Astronautics, Stanford University, Stanford, Calif., SUDAAR 482, Aug. 1974.

[12]Ashkenazi, A., "Parameter Insensitive Controllers," Department of Aeronautics and Astronautics, Stanford University, Stanford, Calif., SUDAAR 520, June 1980.

[13]Martin, G.D. and Bryson, A.E., "Attitude Control of a Flexible Spacecraft," *Journal of Guidance and Control,* Vol. 3, Jan.-Feb. 1980, pp. 37-41.